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The Odd Eight-Vertex Model

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We consider a vertex model on the simple-quartic lattice defined by line graphs on the lattice for which there is always an odd number of lines incident at a vertex. This is the odd 8-vertex model which has eight possible vertex configurations. We establish that the odd 8-vertex model is equivalent to a staggered 8-vertex model. Using this equivalence we deduce the solution of the odd 8-vertex model when the weights satisfy a free-fermion condition. It is found that the free-fermion model exhibits no phase transitions in the regime of positive vertex weights. We also establish the complete equivalence of the free-fermion odd 8-vertex model with the free-fermion 8-vertex model solved by Fan and Wu. Our analysis leads to several Ising model representations of the free-fermion model with pure 2-spin interactions.

KEY WORDS: Odd eight-vertex model; free-fermion model; exact solution.

1. INTRODUCTION

In a seminal work which opened the door to a new era of exactly solvable models in statistical mechanics, Lieb[1, 2] in 1967 solved the problem of the residual entropy of the square ice. His work led soon thereafter to the solution of a host of more general lattice models of phase transitions. These include the five-vertex model,[3, 4] the F model,[5] the KDP model,[6] the general six-vertex model,[7] the free-fermion model solved by Fan and Wu,[8] and the symmetric 8-vertex model solved by Baxter.[9] All these previously considered models are described by line graphs drawn on a simple-quartic lattice where the number of lines incident at each vertex is even, and therefore can be regarded as the "even" vertex models.

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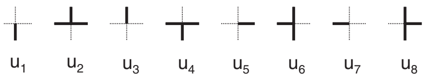

Fig. 1. Vertex configurations of the odd 8-vertex model and the associated weights.

In this paper we consider the *odd* vertex models, a problem that does not seem to have attracted much past attention. Again, one draws line graphs on the simple-quartic lattice but with the restriction that the number of lines incident at a vertex is always odd. There are again eight possible ways of drawing lines at a vertex, and this leads to the *odd 8-vertex model.* Besides being a challenging mathematical problem by itself, as we shall see the odd 8-vertex model includes some well-known unsolved lattice-statistical problems. It also finds applications in enumerating dimer configurations.[10]

Consider a simple-quartic lattice of N vertices and draw lines on the lattice such that the number of lines incident at a vertex is always odd, namely, 1 or 3. There are eight possible vertex configurations which are shown in Fig. 1. To vertices of type i $(=1, 2,..., 8)$ we associate weights $u_i > 0$. Our goal is to compute the partition function

$$Z_{12\cdots 8} \equiv Z(u_1, u_2,..., u_8) = \sum_{\text{o.l.g.}} u_1^{n_1} u_2^{n_2} \cdots u_8^{n_8} \tag{1}$$

where the summation is taken over all aforementioned odd line graphs, and n_i is the number of vertices of the type (i). The per-site "free energy" is then computed as

$$\psi = \lim_{N \to \infty} \frac{1}{N} \ln Z_{12\cdots 8}. \tag{2}$$

The partition function (1) possesses obvious symmetries. An edge can either have a line or be vacant. By reversing the line-vacancy role one obtains the symmetry

$$Z_{12345678} = Z_{21436587}. \tag{3}$$

Similarly, the left-right and up-down symmetries dictate the equivalences

$$Z_{12345678} = Z_{12347856} = Z_{34125678}, \tag{4}$$

and successive 90° counter-clockwise rotations of the lattice lead to

$$Z_{12345678} = Z_{78561243} = Z_{34127856} = Z_{56783421}. \tag{5}$$

These are intrinsic symmetries of the odd 8-vertex model.

The odd 8-vertex model encompasses an unsolved Ashkin–Teller model[11] as a special case (see below). It also generates other known solutions. For example, it is clear from Fig. 1 that by taking

$$\begin{aligned} u_1 &= y, & u_3 &= 1 \\ u_5 &= x, & u_7 &= 1 \\ u_2 = u_4 &= u_6 = u_8 = 0 \end{aligned} \tag{6}$$

(and assuming periodic boundary conditions) the line graphs generate close-packed dimer configurations on the simple-quartic lattice with activities x and y. The solution of (1) in this case is well-known.[12, 13]

2. EQUIVALENCE WITH A STAGGERED VERTEX MODEL

Our approach to the odd 8-vertex model is to explore its equivalence with a staggered 8-vertex model. We first recall the definition of a staggered 8-vertex model.[14]

A staggered 8-vertex model is an (even) 8-vertex model with sublattice-dependent vertex weights. It is defined by 16 vertex weights $\{\omega_i\}$ and $\{\omega'_i\}$, $i = 1, 2,..., 8$, one for each sublattice, associated with the 8 (even) line graph configurations shown in Fig. 2.

The partition function of the staggered 8-vertex model is

$$Z_{\text{stag}}(\omega_1, \omega_2,..., \omega_8; \omega'_1, \omega'_2,..., \omega'_8) = \sum_{\text{e.l.g.}} \prod_{i=1}^{8} [\omega_i{}^{n_i}(\omega'_i)^{n'_i}] \tag{7}$$

where the summation is taken over all even line graphs, and n_i and n'_i are, respectively, the numbers of vertices with weights ω_i and ω'_i. It is convenient to abbreviate the partition function by writing

$$Z_{\text{stag}}(\omega_1, \omega_2,..., \omega_8; \omega'_1, \omega'_2,..., \omega'_8) \equiv Z_{\text{stag}}(12345678; 1'2'3'4'5'6'7'8'). \tag{8}$$

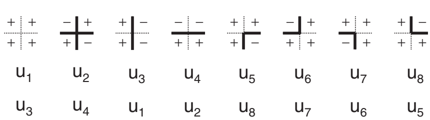

Fig. 2. An equivalent staggered 8-vertex model and the associated spin configurations on the dual.

When $\omega_i = \omega'_i$ for all i, the staggered 8-vertex model reduces to the usual 8-vertex model with uniform weights, which remains unsolved for general ω_i. When $\omega_i \neq \omega'_i$ the problem is obviously even harder. The consideration of the sublattice symmetry implies that we have

$$Z_{\text{stag}}(12345678;\ 1'2'3'4'5'6'7'8') = Z_{\text{stag}}(1'2'3'4'5'6'7'8';\ 12345678). \tag{9}$$

Returning to the odd 8-vertex model we have the following result:

Theorem. The odd 8-vertex model (1) is equivalent to a staggered 8-vertex model (8) with the equivalence

$$\begin{aligned} Z_{12\cdots 8} &= Z_{\text{stag}}(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8;\ u_3, u_4, u_1, u_2, u_8, u_7, u_6, u_5) \\ &= Z_{\text{stag}}(u_5, u_6, u_8, u_7, u_1, u_2, u_3, u_4;\ u_7, u_8, u_6, u_5, u_4, u_3, u_1, u_2), \end{aligned}$$

or, in abbreviations,

$$\begin{aligned} Z_{12\cdots 8} &= Z_{\text{stag}}(12345678;\ 34128765) \\ &= Z_{\text{stag}}(56871243;\ 78654312). \end{aligned} \tag{10}$$

Proof. Let A and B be the two sublattices each having $N/2$ sites. Consider the set S of $N/2$ edges each of which connecting an A site to a B site immediately below it. By reversing the roles of occupation and vacancy on these edges, the vertex configurations of Fig. 1 are converted into configurations with an even number of incident lines. Because of the particular choice of S, however, the vertex weights are sublattice-dependent and we have a *staggered* 8-vertex model.

For sites on sublattice A, the conversion maps a vertex type (i) in Fig. 1 into a type (i) in Fig. 2 so that $\omega_i = u_i$ for *all* i on A. At B sites the conversion maps type (3) in Fig. 1 to type (1) in Fig. 2, (4) to (2) with $\omega'_1 = u_3$, $\omega'_2 = u_4$, etc. Writing compactly and rearranging the B weights according to configurations in Fig. 2, the mappings are

$$\begin{aligned} \omega\{12345678\} &\to u\{12345678\}, \qquad \text{at } A \text{ sites} \\ \omega'\{12345678\} &\to u\{34128765\}, \qquad \text{at } B \text{ sites.} \end{aligned} \tag{11}$$

This establishes the first line in (10).

The line-vacancy conversion can also be carried out for any of the three other edge sets connecting every A site to the B site above it, on the right, or on the left. It is readily verified that these considerations lead to the equivalence given by the second line in (10), and two others obtained from (10) by applying the sublattice symmetry (9). ∎

Remark. Further equivalences can be obtained by combining (3)–(5) with the sublattice symmetry (9).

The special case of

$$\begin{aligned} u_1 &= u_2 = u_3 = u_4 \\ u_5 &= u_6, \quad u_7 = u_8 \end{aligned} \tag{12}$$

is an Ashkin–Teller model as formulated in ref. 15 which remains unsolved. Another special case is when the weights satisfy

$$u_1 u_2 + u_3 u_4 = u_5 u_6 + u_7 u_8. \tag{13}$$

Then from (10) the staggered 8-vertex model weights satisfy the free-fermion condition

$$\begin{aligned} \omega_1\omega_2 + \omega_3\omega_4 &= \omega_5\omega_6 + \omega_7\omega_8 \\ \omega'_1\omega'_2 + \omega'_3\omega'_4 &= \omega'_5\omega'_6 + \omega'_7\omega'_8 \end{aligned} \tag{14}$$

for which the solution has been obtained in ref. 14. This case is discussed in the next section.

3. THE FREE-FERMION SOLUTION

In this section we consider the odd 8-vertex model (1) satisfying the free-fermion condition (13). In the language of the first line of the equivalence (10) we have the staggered vertex weights

$$\begin{aligned} \omega_1 &= \omega'_3 = u_1, \qquad & \omega_2 &= \omega'_4 = u_2 \\ \omega_3 &= \omega'_1 = u_3, \qquad & \omega_4 &= \omega'_2 = u_4 \\ \omega_5 &= \omega'_7 = u_5, \qquad & \omega_6 &= \omega'_8 = u_6 \\ \omega_7 &= \omega'_5 = u_1, \qquad & \omega_8 &= \omega'_6 = u_8, \end{aligned} \tag{15}$$

and hence the condition (14) is satisfied. This leads to the free-fermion staggered 8-vertex model studied in ref. 14. Using results of ref. 14 and the weights (15), we obtain after a little reduction the solution

$$\psi = \frac{1}{16\pi^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \ln F(\theta, \phi) \tag{16}$$

where

$$F(\theta, \phi) = 2A + 2D \cos(\theta - \phi) + 2E \cos(\theta + \phi) + 4\Delta_1 \sin^2 \phi + 4\Delta_2 \sin^2 \theta \tag{17}$$

with

$$\begin{aligned}
A &= (u_1 u_3 + u_2 u_4)^2 + (u_5 u_7 + u_6 u_8)^2 \\
D &= (u_5 u_7)^2 + (u_6 u_8)^2 - 2u_1 u_2 u_3 u_4 \\
E &= -(u_1 u_3)^2 - (u_2 u_4)^2 + 2u_5 u_6 u_7 u_8 \\
\Delta_1 &= (u_1 u_2 - u_5 u_6)^2 > 0 \\
\Delta_2 &= (u_3 u_4 - u_5 u_6)^2 > 0.
\end{aligned} \tag{18}$$

As an example, specializing (16) to the weights (6) for the dimer problem, we have $A = x^2 + y^2$, $D = x^2$, $E = -y^2$, $\Delta_1 = \Delta_2 = 0$, and (16) leads to the known dimer solution[12, 13]

$$\psi_{\text{dimer}} = \frac{1}{\pi^2} \int_0^{\pi/2} d\omega \int_0^{\pi/2} d\omega' \ln(4x^2 \sin^2 \omega + 4y^2 \sin^2 \omega'), \tag{19}$$

which has no phase transitions. More generally for $u_i > 0$ we have $A > |D| + |E|$ and hence

$$F(\theta, \phi) > 0.$$

As a result, the free energy ψ given by (16) is analytic and there is no singularity in ψ implying that the odd free-fermion 8-vertex model has no phase transition.

4. EQUIVALENCE WITH THE FREE-FERMION MODEL OF FAN AND WU

The free energy (16) is of the form of that of the free-fermion model solved by Fan and Wu.[8] To see this we change integration variables in (16) to

$$\alpha = \theta + \phi, \qquad \beta = \theta - \phi, \tag{20}$$

the expression (16) then assumes the form

$$\begin{aligned}
\psi = \frac{1}{16\pi^2} \int_0^{2\pi} d\alpha \int_0^{2\pi} d\beta \ln[&2A_1 + 2E \cos \alpha + 2D \cos \beta \\
&- 2\Delta_1 \cos(\alpha - \beta) - 2\Delta_2 \cos(\alpha + \beta)]
\end{aligned} \tag{21}$$

where, after making use of (13),

$$\begin{aligned} A_1 &= A + \Delta_1 + \Delta_2 \\ &= (u_1u_2 + u_3u_4)^2 + (u_1u_3)^2 + (u_2u_4)^2 + (u_5u_7)^2 + (u_6u_8)^2. \end{aligned}$$

Comparing (21) with Eq. (16) of ref. 8, we find

$$\psi = \psi_{\mathrm{FF}}/2 \tag{22}$$

where ψ_{FF} is the per-site free energy of an 8-vertex model with *uniform* weights w_1, w_2,..., w_8 satisfying the free-fermion condition

$$w_1w_2 + w_3w_4 = w_5w_6 + w_7w_8 \tag{23}$$

and

$$\begin{aligned} A_1 &= (w_1^2 + w_2^2 + w_3^2 + w_4^2)/2 \\ D &= w_1w_4 - w_2w_3 \\ E &= w_1w_3 - w_2w_4 \\ \Delta_1 &= w_1w_2 - w_5w_6 \\ \Delta_2 &= w_5w_6 - w_3w_4. \end{aligned} \tag{24}$$

We can solve for w_1, w_2, w_3, w_4, and w_5w_6 from the five equations in (24), and then determine w_7w_8 from (23).

By equating (24) with (18), it can be verified that one has

$$\begin{aligned} (-w_1 + w_2 + w_3 + w_4)^2 &= 2(A_1 - D - E - \Delta_1 - \Delta_2) = v_1^2 \\ (w_1 - w_2 + w_3 + w_4)^2 &= 2(A_1 + D + E - \Delta_1 - \Delta_2) = v_2^2 \\ (w_1 + w_2 - w_3 + w_4)^2 &= 2(A_1 + D - E + \Delta_1 + \Delta_2) = v_3^2 \\ (w_1 + w_2 + w_3 - w_4)^2 &= 2(A_1 - D + E + \Delta_1 + \Delta_2) = v_4^2, \end{aligned} \tag{25}$$

where[3]

$$\begin{aligned} v_1 &= 2(u_1u_3 + u_2u_4) \\ v_2 &= 2(u_5u_7 + u_6u_8) \\ v_3 &= 2\sqrt{(u_1u_2 + u_3u_4)^2 + (u_1u_3 - u_2u_4)^2 + (u_5u_7 - u_6u_8)^2} \\ v_4 &= 2(u_1u_2 + u_3u_4). \end{aligned} \tag{26}$$

[3] The apparent asymmetry in the expression of v_3 can be traced to the choice of the edge set S used in Section 2 in deducing the equivalent staggered 8-vertex model.

Then, taking the square root of (25), one obtains the explicit solution

$$w_i = (v_1 + v_2 + v_3 + v_4 - 2v_i)/4, \qquad i = 1, 2, 3, 4. \tag{27}$$

The 4th line of (24) now yields

$$w_5 w_6 = w_1 w_2 - (u_1 u_2 - u_5 u_6)^2, \tag{28}$$

and $w_7 w_8$ is obtained from (23).

The free-fermion model is known[8] to be critical at

$$2w_i = w_1 + w_2 + w_3 + w_4, \qquad i = 1, 2, 3, 4 \tag{29}$$

which is equivalent to $v_i = 0$. It is then clear from (26) that the critical point (29) lies outside the region $u_i > 0$ and this confirms our earlier conclusion that the free-fermion odd 8-vertex model does not exhibit a transition in the regime of positive weights. Our results also show that the model with some $u_i = 0$, e.g., $u_7 = u_8 = 0$, is critical. This is reminiscent to the known fact of the even vertex models that the 8-vertex model is critical in the 6-vertex model subspace.

Finally, we point out that the equivalence with a free-fermion model described in this section is based on the comparison of the free energies of the two models in the thermodynamic limit. It remains to be seen whether a mapping can be established which leads to (27) directly, and thus the word "equivalence" is used in a weaker sense.

5. ISING REPRESENTATIONS OF THE FREE-FERMION MODEL

The free-fermion odd 8-vertex model can be formulated as Ising models with pure 2-spin interactions in several different ways. In the preceding section we have established its equivalence with the Fan–Wu free-fermion model. Baxter[16] has shown that the Fan–Wu free-fermion model is equivalent to a checkerboard Ising model and that asymptotically it can be decomposed into four overlapping Ising models. It follows that the odd 8-vertex model possesses the same properties, namely, it is equivalent to a checkerboard Ising model and can be similarly decomposed asymptotically. We refer to ref. 16 for details of analysis.

An *alternate* Ising representation can be constructed as follows: Consider the equivalent staggered 8-vertex model given in the first line of (10). We place Ising spins on dual lattice sites as shown in Fig. 2 and write the partition function as

$$Z_{\text{Ising}} = \sum_{\text{spin config.}} \prod_{A} W(a, b, c, d) \prod_{B} W'(a, b, c, d) \tag{30}$$

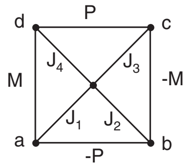

Fig. 3. Ising interactions in $W(a, b, c, d)$.

where the summation is taken over all spin configurations, and W and W' are, respectively, the Ising Boltzmann factors associated with four spins $a, b, c, d = \pm 1$ surrounding each A and B sites. Since the vertex to spin configuration mapping is $1:2$, we have the equivalence

$$Z_{12\cdots 8} = Z_{\text{Ising}}/2. \tag{31}$$

We next require the Ising Boltzmann factors W and W' to reproduce the vertex weights ω and ω' in (10). Now to each vertex in the free-fermion model there are six independent parameters after taking into account the free-fermion condition (13) and an overall constant. We therefore need six Ising parameters which we introduce as interactions shown in Fig. 3 for $W(a, b, c, d)$ on sublattice A. Namely, we write

$$W(a, b, c, d) = 2\rho e^{M(ad-bc)/2+P(cd-ab)/2} \cosh(J_1 a + J_2 b + J_3 c + J_4 d) \tag{32}$$

where ρ is an overall constant. Explicitly, a perusal of Fig. 2 leads to the expressions

$$\begin{aligned}
u_1 &= 2\rho \cosh(J_1 + J_2 + J_3 + J_4), & u_2 &= 2\rho \cosh(J_1 - J_2 + J_3 - J_4)\\
u_3 &= 2\rho \cosh(J_1 - J_2 - J_3 + J_4), & u_4 &= 2\rho \cosh(J_1 + J_2 - J_3 - J_4)\\
u_5 &= 2\rho e^{M+P} \cosh(J_1 - J_2 + J_3 + J_4), & u_6 &= 2\rho e^{-M-P} \cosh(J_1 + J_2 + J_3 - J_4)\\
u_7 &= 2\rho e^{P-M} \cosh(-J_1 + J_2 + J_3 + J_4), & u_8 &= 2\rho e^{M-P} \cosh(J_1 + J_2 - J_3 + J_4).
\end{aligned} \tag{33}$$

These weights satisfy the free-fermion condition (13) automatically.[4]

Equation (33) can be used to solve for J_1, J_2, J_3, J_4, M, P and the overall constant ρ in terms of the weights u_i. First, using the first four

[4] Expressions in Eq. (33) are the same as Eq. (2.5) in ref. 16 except the interchange of expressions u_7 and u_8 due to the different ordering of configurations (7) and (8).

equations one solves for J_1, J_2, J_3, J_4 in terms of $\cosh^{-1}(u_i/2\rho)$, $i=1, 2, 3, 4$. Then the overall constant ρ is solved from the equation

$$\frac{u_5 u_6}{u_7 u_8} = \frac{\cosh 2(J_1+J_3)+\cosh 2(J_2-J_4)}{\cosh 2(J_1-J_3)+\cosh 2(J_2+J_4)} \tag{34}$$

and M, P are given by

$$\begin{aligned} e^{4M} &= \left(\frac{u_5 u_8}{u_6 u_7}\right)\left[\frac{\cosh 2(J_1-J_4)+\cosh 2(J_2+J_3)}{\cosh 2(J_1+J_4)+\cosh 2(J_2-J_3)}\right], \\ e^{4P} &= \left(\frac{u_5 u_7}{u_6 u_8}\right)\left[\frac{\cosh 2(J_1-J_2)+\cosh 2(J_3+J_4)}{\cosh 2(J_1+J_2)+\cosh 2(J_3-J_4)}\right]. \end{aligned} \tag{35}$$

For B sites, we note that the weights are precisely those of A sites with the interchanges $u_1 \leftrightarrow u_3$, $u_2 \leftrightarrow u_4$, $u_5 \leftrightarrow u_8$, $u_6 \leftrightarrow u_7$. In terms of the spin configurations, these interchanges correspond to the negation of the spins b and c. Thus we have

$$\begin{aligned} W'(a, b, c, d) &= W(a, -b, -c, d) \\ &= 2\rho e^{M(ad-bc)/2-P(cd-ab)/2} \cosh(J_1 a - J_2 b - J_3 c + J_4 d). \end{aligned} \tag{36}$$

This Boltzmann factor is the same as (30) with the same J_1, J_4, M, ρ and the negation of J_2, J_3, and P. Namely, we have

$$\begin{aligned} &J_1' = J_1, \qquad J_2' = -J_2, \qquad M' = M, \qquad \rho' = \rho \\ &J_3' = -J_3, \qquad J_4' = J_4, \qquad P' = -P \end{aligned} \tag{37}$$

Putting the Ising interactions together, interactions M and M' cancel and we obtain the Ising representation shown in Fig. 4. The Ising model now has five independent variables J_1, J_2, J_3, J_4, and $2P$.

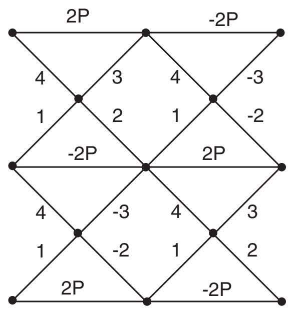

Fig. 4. An Ising model representation of the odd 8-vertex model. The number -2 stands for $-J_2$, etc.

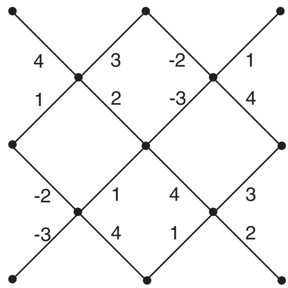

Fig. 5. An Ising model representation of the odd 8-vertex model when $u_5 = u_6$, $u_7 = u_8$. The number -2 stands for $-J_2$, etc.

If we have further

$$u_5 = u_7, \qquad u_6 = u_8, \tag{38}$$

then from the configurations in Fig. 2, we see that the weights now possess an additional up-down symmetry, namely,

$$W(a, b, c, d) = W(d, c, a, b). \tag{39}$$

Consequently we have $P = -P$ implying $P = 0$. The Ising model representation is then of the form of a simple-quartic lattice with staggered interactions as shown in Fig. 4 with $P = 0$.

If we have

$$u_5 = u_6, \qquad u_7 = u_8, \tag{40}$$

it can be seen from Fig. 2 that the A weights have the symmetry

$$W(a, b, c, d) = W(c, d, b, a) \tag{41}$$

and for B sites we have

$$W'(a, b, c, d) = W(-c, d-b, a). \tag{42}$$

In the resulting Ising model both M and P now cancel and the lattice is shown in Fig. 5.

6. SUMMARY

We have introduced an odd 8-vertex model for the simple-quartic lattice and established its equivalence with a staggered 8-vertex model. We showed that in the free-fermion case the odd 8-vertex model is completely equivalent to the free-fermion model of Fan and Wu in a noncritical regime. Several Ising model representations of the free-fermion odd 8-vertex model are also deduced.

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